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Simple Groups with Cyclic Central 2-Sylow Intersections

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1. INTRODUCTION

In this paper a group means a finite group and a simple group means a non-abelian finite simple group. A subgroup D of a group G is called a 2-*Sylow intersection* if there exist distinct 2-Sylow subgroups S_1 and S_2 of G such that $D = S_1 \cap S_2$. An involution of G is called *central* if it is contained in a center of a 2-Sylow subgroup of G. A 2-Sylow intersection is called *central* if it contains a central involution or it is trivial.

In 1964, Suzuki [7] determined all simple groups with all 2-Sylow intersections being trivial. Using a recent fusion theorem by Shult [2, p. 62] the author proved [3] that no additional simple groups are involved if Suzuki's condition is weakened to read: all central 2-Sylow intersections are trivial.

In 1971 Mazurow [5] determined all simple groups G which satisfy the following condition:

(A) All 2-Sylow intersections of G are cyclic.

The aim of this work is to prove that no additional simple groups are involved if condition A is weakened to read:

(A*) all central 2-Sylow intersections of G are cyclic.

THEOREM A. *Let G be a simple group satisfying condition* A*. *Then G is isomorphic to one of the following groups*:

(i) $PSL(2, q)$, $q = 2^n > 2$;

(ii) $Sz(q)$, $q = 2^n \geqslant 8$;

(iii) $PSU(3, q)$, $q = 2^n > 2$;

(iv) $PSL(2, q)$, $q \equiv 3$ *or* $5 (\mathrm{mod}\ 8)$, $q > 5$ *and*

(v) J, *the Janko group.*

The proof of Theorem A is independent of [5]. It was shown in [4] that if G is a simple group satisfying condition A* and if the centralizer of a central involution of G is solvable, then G is of type (i)–(iv). Thus in order to prove Theorem A it suffices to prove the following characterization of the Janko group J:

THEOREM B. *Let G be a simple group satisfying condition* A* *and suppose that the centralizer of a central involution of G is nonsolvable. Then $G \cong J$.*

In this paper $G \in A$ or $G \in A^*$ means that the group G satisfies condition A or A*, respectively. A group is called of type (x) if it is isomorphic to a group mentioned in Theorem A under (x). An involution z of the group G is called *isolated* if a Sylow group containing it has no elements which are conjugate to z in G and not equal to z. The maximal normal 2-subgroup and 2′-subgroup of G are denoted, respectively, by $O_2(G)$ and $O(G)$.

2. Proof of Theorem B

Let G be a counterexample of minimal order and let $M = \{N_G(V) \mid V$ is a 2-subgroup of G containing a central involution, $N_G(V)$ is nonsolvable and $|N_G(V)|_2 = |G|_2\}$. By our assumptions M is not empty. Let

$$N = \{H \mid H \in M, \ |O_2(H)| \geqslant |O_2(H_1)| \text{ for all } H_1 \in M\}$$

and finally let H be a fixed element of N^*, the set of maximal elements of N.

Denote by S the maximal normal solvable subgroup of H and let V be an S_2-subgroup of S. Since $H \in M$, V contains a central involution of G.

A series of lemmas will lead us to a contradiction.

LEMMA 1. $H = N_G(V)$ *and* $V = O_2(H)$. *Thus V is cyclic and* $S = V \times O(H)$.

Proof. By the Frattini argument $H = SN_H(V)$. Hence $N_G(V)$ is nonsolvable and as $|S|_2 = |N_H(V) \cap S|_2$, it follows that $|N_H(V)|_2 = |H|_2$ and $N_G(V) \in M$. Since $H \in N$ and $O_2(N_G(V)) \supseteq V$, we must have $V = O_2(H)$ and as $H \in N^*$, it follows that $H = N_G(V)$. Since H is non-2-closed and $H \in A^*$, V is cyclic and by Burnside's theorem $S = V \times O(H)$.

LEMMA 2. $\bar{H} = H/S$ *has cyclic* 2-*Sylow intersections. In particular,* $\bar{H}$ *satisfies condition* A*.

Proof. Let $(P/S) \cap (P_1/S) = D/S$ be a 2-Sylow intersection of $\bar{H}$. Then there exist 2-Sylow subgroups Q and Q_1 of G such that $P = O(H)Q$ and

$P_1 = O(H)Q_1$. Similarly $D = O(H)D_1$, where D_1 is a 2-subgroup of G, and there exist $a, a_1 \in O(H)$ such that

$$V \subseteq D_1 \subseteq Q^a \cap Q_1^{a_1}.$$

As $P \neq P_1$ also $Q^a \neq Q_1^{a_1}$ and since $G \in A^*$, D_1 is cyclic and so is D/S.

LEMMA 3. *Let $\bar{L}$ be the product of all minimal normal subgroups of $\bar{H}$. Then $\bar{L}$ is of type* (i)–(v).

Proof. $\bar{L}$ is a direct product of non-abelian simple groups. Since by Lemma 2 the 2-Sylow intersections of $\bar{L}$ are cyclic, $\bar{L}$ is a simple group.

If a centralizer of a central involution of $\bar{L}$ is solvable, then it follows by [4] that $\bar{L}$ is of type (i)–(iv). Otherwise, it follows by induction that $\bar{L} \simeq J$.

LEMMA 4. *$\bar{H}/\bar{L}$ is of odd order.*

Proof. Since by Lemma 3 $\bar{L}$ is the unique minimal normal subgroup of $\bar{H}$, we have $C_{\bar{H}}(\bar{L}) = 1$. If $\bar{L}$ is of type (ii), then $\bar{H}/\bar{L}$ is of odd order by Theorem 11 in [6].

Suppose that $\bar{H}/\bar{L}$ is of even order and let $\bar{H}_0$ be a subgroup of $\bar{H}$ such that $|\bar{H}_0 : \bar{L}| = 2$. Let $A \in \mathrm{Syl}_2(\bar{L})$; since A is noncyclic, it follows by Lemma 2 that $N_{\bar{H}_0}(A)$ is 2-closed. Let $B \in \mathrm{Syl}_2(N_{\bar{H}_0}(A))$; since $|\bar{H}_0 : \bar{L}| = 2$, $B \in \mathrm{Syl}_2(\bar{H}_0)$. We will proceed with a series of steps.

(a) $N_{\bar{H}_0}(A) = N_{\bar{H}_0}(B)$.

Pf. As $N_{\bar{H}_0}(A)$ is 2-closed, $N_{\bar{H}_0}(A) \subseteq N_{\bar{H}_0}(B)$. But $B \cap \bar{L} = A$ and $\bar{L} \lhd \bar{H}_0$, so $N_{\bar{H}_0}(B) \subseteq N_{\bar{H}_0}(A)$.

(b) Let C be a 2-complement of $N_{\bar{H}_0}(B)$. Then $N_{\bar{L}}(A) = AC$ and $[B, C] = A$.

Pf. It follows by (a) that $N_{\bar{L}}(A) = AC$. As $\bar{L}$ is of type (i)–(v), $C_A(C) = 1$ and $A = [A, C]$. Since $[B : A] = 2$ and A is C-invariant, it follows that $[B, C] \subseteq A$ and consequently $[B, C] = A$.

(c) There exists an involution x in $C_B(C)$ and $C_A(x) \neq 1$.

Pf. As by (b) $B = C_B(C)[B, C] = C_B(C)A$, $C_B(C) \neq 1$. Since A is normal in B, $C_A(x) \neq 1$.

(d) $C_A(x) = A$ and $B = \langle x \rangle A$.

Pf. Let $A_0 = C_A(x)$; by (c) A_0 is a nontrivial C-invariant subgroup of A. If $\bar{L}$ is of type (i), (iv) or (v) then A is the only nontrivial C-invariant subgroup of A, hence $A = A_0$. If $\bar{L}$ is of type (iii) then also $Z(A)$ is a C-invariant subgroup of A. Thus it suffices to prove that $A_0 \neq Z(A)$. Suppose that $A_0 = Z(A)$ and consider $A_1 = [\langle x \rangle, A]$. Then A_1 is a nontrivial C-invariant subgroup of A and since $A_1 \subseteq B' \cap A \subset A$, $A_1 = Z(A)$. It is easy to check that $A_1 = \{[x, a] \mid a \in A\}$ and $[x, a] = [x, b]$ for some $a, b \in A$ if and only if $ab^{-1} \in C_A(x) = Z(A)$. Thus $|Z(A)| = |A_1| = |A : Z(A)|$ in contradiction to the fact that $|A| = |Z(A)|^3$.

Since $C_A(C) = 1$, $x \notin A$ and $B = \langle x \rangle A$.

(e) A contradiction.

Pf. By (d) $x \in Z(A\langle x \rangle C) = Z(N_{\bar{H}_0}(B))$. Since by Lemma 2 $\bar{H}_0$ has cyclic 2-Sylow intersections, it follows by Lemma 3 in [4] and Burnside's theorem that x is an isolated involution in $\bar{H}_0$. Glauberman's theorem [1] then implies that $x \in Z(\bar{H}_0)$, as $O(\bar{H}_0) = 1$. But then $x \in C_{\bar{H}}(\bar{L}) = 1$, a contradiction. The proof of Lemma 4 is complete.

Let R be a fixed S_2-subgroup of H. Let z be the involution of V. Obviously $z \in Z(R)$ and $R \in \mathrm{Syl}_2(G)$. We have shown in Lemma 4 that $\bar{H} = H/O(H) \times V$ contains a subgroup of odd index, which is of type (i)–(v). Thus R/V is isomorphic to an S_2-subgroup of a group of type (i)–(v).

Since z is a central involution in G, it follows by Lemma 3 in [4] that all conjugates of z in G belonging to R are central and can be obtained by conjugation in $N_G(R)$. Since G is simple, z is not isolated and consequently there exists $x \in N_G(R) - C_G(z)$. Let x be a fixed such element and let $u = z^x$. Then u is a central involution in R, different from z.

LEMMA 5. $V = \langle z \rangle$.

Proof. Let $V_0 = \langle z \rangle$. Since $u \neq z$, $V_0^x \cap V_0 = 1$ and consequently $V^x \cap V = 1$. As $V^x \lhd R$, V^x is isomorphic to a normal cyclic subgroup of R/V. However, the S_2-subgroups of groups of type (i)–(v) have no cyclic normal subgroups of order larger than two. Consequently $|V| = 2$.

LEMMA 6. *$H/\langle z \rangle$ has one conjugate class of involutions.*

Proof. Since $\bar{L}$ is of type (i)–(v), it has one conjugate class of involutions. As $|\bar{H} : \bar{L}|$ is odd, so does $\bar{H}$. But

$$\bar{H} = H/O(H)V \cong (H/V)/(O(H)\, V/V)$$

and $O(H)\, V/V$ is of odd order, so also H/V has one conjugate class of involutions, as required.

LEMMA 7. *If $y\langle x \rangle$ is an involution in $H/\langle z \rangle$ then y is conjugate in H either to u or to uz. Consequently,*

(a) *y is an involution*;

(b) *G has at most two conjugate classes of involutions*;

(c) *every involution in G is central and*

(d) $Z(R) \supseteq \Omega_1(R)$.

Proof. Since $H/\langle z\rangle$ has one conjugate class of involutions, there exists $t \in H \subseteq C_G(z)$ such that $y\langle z\rangle = u^t\langle z\rangle$. Thus either $y = u^t$ or $y = u^t z = (uz)^t$ and in particular y is an involution. As $| G : H |$ is odd and z is conjugate to u, (b) follows.

Since every involution in G is conjugate either to u or to uz, every involution in G is central and by Lemma 3 in [4] every involution in R belongs to $Z(R)$, hence $Z(R) \supseteq \Omega_1(R)$.

LEMMA 8. *Either R is elementary abelian or G has 2 conjugate classes of involutions.*

Proof. Suppose that R is not elementary abelian and let w be an element of R of order 4. Then by Lemma 7 w^2 is a central involution in R. Suppose that z is conjugate to w^2 in G; then there exists $t \in N_G(R)$ such that $z = (w^2)^t = (w^t)^2$ and by Lemma 7 w^t is an involution, a contradiction. Thus z and w^2 belong to distinct conjugate classes in G and again by Lemma 7 G has exactly 2 conjugate classes of involutions.

LEMMA 9. *R is not elementary abelian.*

Proof. Since the centralizer of an involution in G is nonsolvable and the Ree-type-groups do not satisfy condition A*, it follows by the classification theorem of Walter [8] that $G \cong J$, in contradiction to G being a counterexample.

LEMMA 10. *G does not have 2 conjugate classes of involutions.*

Proof. Let $|\Omega_1(R)| = q$; then by Lemmas 7–9 R contains $q - 1$ involutions distributed between two conjugate classes of involutions of G. Thus one conjugate class of involutions in G, say K_1, contains an even number of involutions of R, say $2k$ of them. Consequently, the number of elements in K_1 is $| G : N_G(R)| \, 2k/r$, where r is the number of 2-Sylow subgroups of G containing a fixed element x of K_1. As by Lemma 7 x is a central involution of G, $| K_1 |$ is odd, forcing r to be an even integer. However, by Lemma 3 in [4] x belongs to the center of each 2-Sylow subgroup of G which contains it, hence r is the number of 2-Sylow subgroups of $C_G(x)$, which is odd by the Sylow theorem. This contradiction proves Lemma 10.

Theorem B follows immediately from Lemmas 8–10.

References

1. G. Glauberman, Central elements of core-free groups, *J. Algebra* **4** (1966), 403–420.
2. G. Glauberman, Global and local properties of finite groups, *in* "Finite Simple Groups," Academic Press, 1971.
3. M. Herzog, *On 2-Sylow intersections, Israel J. Math.* **11** (1972), 326–327.
4. M. Herzog, *Central 2-Sylow intersections, Pacific J. Math.*, to appear.
5. V. D. Mazurow, Finite simple group with cyclic intersections of Sylow 2-subgroups, *Algebra i Logika* **10** (1971), 188–198, in Russian.
6. M. Suzuki, On a class of doubly transitive groups, *Ann. of Math.* **75** (1962), 105–145.
7. M. Suzuki, Finite groups of even order in which Sylow 2-groups are independent, *Ann. Math.* **80** (1964), 58–77.
8. J. H. Walter, The characterization of finite groups with abelian Sylow 2-subgroups, *Ann of Math.* **89** (1969), 405–514.